A SOLUTION OF THE PROBLEM OF PLANE-PARALLEL PRESSURE-NONPRESSURE FLOW

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A solution is obtained for a problem of pressure-nonpressure flow on the assumption that there is no seepage from the roof of the bed. G. I. Barenblatt's method of integral relations is used to solve the problem with allowance for seepage. The results show that it is sufficient to confine oneself to the first approximation.

Reference [1] is concerned with the motion of a fluid in a bed with a slightly permeable roof and an impermeable floor. In the case of plane-parallel motion this problem reduces to the solution of the system of equations

$$\mu\frac{\partial h}{\partial t}=k\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)+q_0,\qquad q_0=\frac{k_0}{m_0}(H_0+h_V-m)\qquad (0\leqslant x\leqslant l(t))$$
$$\frac{\partial^2 H}{\partial x^2}-\omega^2(H-H_0)=0,\qquad \omega^2=\frac{k_0}{kmm_0}\qquad (l(t)\leqslant x<\infty)\tag{1}$$

Here h is the ordinate of the free surface; H, the head in the bed; H_0, the head outside the roof of the bed and at $x\to\infty$; h_V is the height of the liquid column corresponding to the pressure in the underpressure zone; μ is the water yield coefficient; k and k_0 are the permeabilities of the bed and the roof; m and m_0 are the thickness of the bed and the roof; $l(t)$ is the moving interface of the zones of pressure and nonpressure flow.

V. I. Pankovskii [1] introduces the new variable $y=x/l$, and then, linearizing and neglecting certain terms in the first equation of system (1), solves the latter for the following conditions:

$$h[l(t),t]=m,\qquad l(0)=0,\qquad q_0=0$$
$$\left(h\frac{\partial h}{\partial x}\right)_{x=l}=m\left(\frac{\partial H}{\partial x}\right)_{x=l},\qquad k\left(h\frac{\partial h}{\partial x}\right)_{x=0}=Q>Q_*=km\omega(H_0+h_V-m)\tag{2}$$

where Q_* is the critical flow; the condition for pressure-nonpressure flow is $Q>Q_*$.

Below we present the exact ($q_0=0$) and approximate solutions of linearized system (1) obtained by the method of integral relations [2]. In this case it is possible to confine oneself to the first approximation, since a comparison of the exact and approximate solutions at $q=0$ gives quite good results.

By linearizing the first equation of system (1) in accordance with the second method, setting $u=h^2$, we obtain

$$\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2},\qquad a=\frac{k\sqrt{u}}{\mu}\approx\frac{km}{\mu}\qquad (q_0=0)\tag{3}$$

Noting that the second equation of the system has the solution

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$$H = H_0 - (H_0 + h_V - m)\, e^{-\omega(x-l)}$$

we rewrite conditons (2) as follows:

$$u\,[l\,(t),\, t] = m^2, \qquad (\partial u / \partial x)_{x=0} = 2Q / k = Q_0$$
$$(\partial u / \partial x)_{x=l} = 2m\omega\,(H_0 + h_V - m) = 2Q_* / k = Q_1 \, . \tag{4}$$

If we set

$$q = \partial u / \partial x, \qquad \xi = x / 2\alpha \sqrt{at}, \qquad l\,(t) = 2\alpha \sqrt{at}$$

then, instead of (3), we obtain

$$\frac{d^2 q}{d\xi^2} + 2\alpha^2 \xi \frac{dq}{d\xi} = 0 \tag{5}$$

where α is a still unknown constant.

In this case we can return to the original function u in accordance with the equation

$$u\,(x, t) = \int_0^x q a x + a \int_0^t \left(\frac{\partial q}{\partial x}\right)_{x=0} dt + m^2 \, . \tag{6}$$

The solution of Eq. (5) with conditions (4) is

$$q\,(\xi) = C \operatorname{erf}(\alpha\xi) + Q_0 \qquad C = \frac{Q_1 - Q_0}{\operatorname{erf}\alpha}, \qquad \operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-y^2} \partial y \, . \tag{7}$$

From (6) we find the unknown function

$$u\,(x, t) = m^2 + 2C\left(\frac{at}{\pi}\right)^{1/2} + C\left\{x \operatorname{erf}\left(\frac{x}{2\sqrt{at}}\right) + 2\left(\frac{at}{\pi}\right)^{1/2}\left[\exp\frac{-x^2}{4at} - 1\right] + Q_0 x\right\} . \tag{8}$$

Hence from the first condition we obtain the following transcendental equation for determining α:

$$\sqrt{\pi}\,\alpha\delta \operatorname{erf}\alpha = (1 - \delta)\ \exp\,(-\alpha^2) \qquad (\delta = Q_* / Q) \, . \tag{9}$$

Solution (8) of Eq. (3) can be represented in the following dimensionless form:

$$P = \frac{k\,(m^2 - u)}{4Q\sqrt{at}} = \Delta i \operatorname{erf} c\lambda - \lambda(1 - \Delta) \tag{10}$$

where

$$i \operatorname{erf} c\lambda = \int_\lambda^\infty \operatorname{erf} c\, y dy, \quad \lambda = \frac{x}{2\sqrt{at}}, \quad \Delta = \frac{1 - \delta}{\operatorname{erf}\alpha} \, .$$

We note that as $\delta \to 0$, $\alpha \to \infty$ and (10) is the solution of the well-known filtration problem.

If seepage from above is taken into account, instead of the first equation of system (1), setting $u = 1 - h^2/m^2 + \varepsilon t/m^2$, we obtain

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \xi^2}, \quad \tau = \frac{k_0}{m_0 \mu} t, \quad \xi = \omega x, \quad \varepsilon = 2\frac{m q_0}{\mu} \, . \tag{11}$$

The solution of this equation is found in the form

$$u(\xi, \tau) = u_0(\tau) + u_1(\tau)(\xi/\xi_0) + u_2(\tau)(\xi/\xi_0)^2 \quad (\xi_0 = \omega l). \tag{12}$$

From condition (2)

$$\left(\frac{\partial u}{\partial \xi}\right)_{\xi=0} = -\frac{2Q}{k\omega m^2} = a_1, \qquad \left(\frac{\partial u}{d\xi}\right)_{\xi=\xi_0} = -\frac{2Q_*}{2\omega m^2} = a_2$$

$$u(\xi_0, \tau) = \varepsilon_0 \tau, \qquad \varepsilon_0 = \varepsilon m_0 \mu / k_0 m^2$$

we find

$$u_0 = \varepsilon_0 \tau - 1/2(a_1 + a_2)\xi_0, \quad u_1 = a_1 \xi_0, \quad u_2 = 1/2(a_2 - a_1)\xi_0. \tag{13}$$

In this case in order to determine the moving boundary $\xi_0(\tau)$ we have the following equation [2]:

$$\frac{d}{d\tau}\int_0^{\xi_0} u\,d\xi = \left(\frac{\partial u}{\partial \xi}\right)_{\xi=\xi_0} - \left(\frac{\partial u}{\partial \xi}\right)_{\xi=0} + u(\xi_0, \tau)\frac{d\xi_0}{d\tau}, \qquad \xi_0(0) = 0$$

or using (13),

$$\frac{2\xi_0 d\xi_0}{\varepsilon_0 \xi_0 + a_1 - a_2} = \frac{3d\tau}{a_2 + {}^1/_2 a_1}. \tag{14}$$

Integrating this equation, we find

$$\xi_0 - (a_1 - a_2)\ln\left(1 + \frac{\varepsilon_0 \xi_0}{a_1 - a_2}\right) = \frac{3\varepsilon_0^2 \tau}{2a_2 + a_1}. \tag{15}$$

From (14) we have $d\xi = 0$ at $\xi_0 = (a_2 - a_1)/\varepsilon_0$; on the other hand, from (15) we have $\xi_0 \to (a_2 - a_1)/\varepsilon_0$ as $\tau \to \infty$ or, in the previous notation, $l \to l(\infty) = (Q - Q_*)/q_0$ as $t \to \infty$.

This shows that as $t \to \infty$ the flow Q as a whole is made up of the lateral inflow Q_* and the inflow from above q_0. Consequently, in the presence of seepage from above, as distinct from the previous case, in which the solution is valid at $h \geq 0$, there is a certain limiting situation corresponding to stationary motion.

Equation (15) can be represented in the following convenient form:

$$\ln(1 - l_0) + l_0 = 3\delta^\circ\tau[(\delta - 1)(1 + 2\delta)]^{-1}, \qquad l_0 = l(t)/l(\infty). \tag{16}$$

When $q_0 = 0$ from (12), (13) and (14) we obtain the following solution:

$$u = -\frac{a_1 + a_2}{2}\xi_0 + a_1\xi_0\left(\frac{\xi}{\xi_0}\right) + \frac{a_2 - a_1}{2}\left(\frac{\xi}{\xi_0}\right)^2, \quad \xi_0^2 = \frac{6(a_1 - a_2)\tau}{2a_2 + a_1}$$

or, substituting the values of a_1 and a_2, carrying out certain transformations and returning to the previous notation,

$$P = \frac{k(m^2 - h^2)}{4Q\sqrt{at}} = \frac{\alpha}{2}\left[1 + \delta - 2\left(\frac{x}{l}\right) + (1 - \delta)\left(\frac{x}{l}\right)^2\right] \tag{17}$$

$$l(t) = 2\alpha\sqrt{at}, \qquad \alpha = \left(1.5\frac{1 - \delta}{1 + 2\delta}\right)^{1/2}. \tag{18}$$

We present values of $\alpha = \alpha'$, calculated from the latter equation and values of the roots ($\alpha = \alpha''$) of transcendental equation (9); a comparison shows that it is sufficient to confine oneself to the first approximation.

$\delta = 0.95$	0.90	0.85	0.80	0.70	0.60	0.50	0.40
$\alpha'' = 0.1608$	0.2315	0.2889	0.3402	0.4344	0.5257	0.6201	0.7233
$\alpha' = 0.1608$	0.2315	0.2887	0.3397	0.4332	0.5222	0.6125	0.7050

In the same way, it is easy to obtain an approximate solution for the isolated pressure bed considered in [3].

LITERATURE CITED

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